CENTAURUS HS VISIT
5/13/10: "Quadratic Equations"

1. If we were to graph a *quadratic equation*, the resulting graph is called a *parabola*. Go ahead and sketch a parabola on an xy-axis. What are the distinguishing features of the graph?

   We want students to say the following things:
   1. the x-intercepts (where it crosses the x-axis)
   2. the y-intercept (where it crosses the y-axis)
   3. the vertex (which is the point of symmetry on the parabola)
   4. whether the parabola opens up or down

   So if we can figure out these four things from the equation we are given, we can do a good job of sketching the parabola. And, in reverse, if you take these features off from your graph, you will be able to find its equation. That is the goal for the following questions we are going to go over with you...

2. Write down the equation $y = x^2 + 6x + 5$. This is the *standard form* of the equation.

   a. Now, *factor* the equation so that it is written as $y = (x + \_\_)(x + \_\_)$. This is the equation given in *factored form.*

      Note: Students should know this already from their last exam and orals session... They can use an "area model" to help them, if needed. $y = (x + 5)(x + 1)$

   b. Why do we like writing the equation in factored form? What is the usefulness?

      (It helps us find the roots, or x-intercepts, of the equation)

      Find the x-intercepts for this parabola. How do we know that these x-values are the x-intercepts?

      (If you visualize the sketch from the first problem, the x-intercepts are where the parabola crosses the x-axis. This happens when $y = 0$*. When* $x = -5$*, the first factor becomes 0, so* $y = 0$*. When* $x = -1$*, the second factor becomes 0, so again,* $y = 0$*.*

   c. Find the y-intercept for this parabola. How do you know that this is the y-intercept?

      (If you visualize a parabola, the y-intercept is where the parabola crosses the y-axis. This happens when $x = 0$*. So we plug* $x = 0$ *into the equation and solve for y, getting* $y = 5$*.)*

   d. Find the *vertex* of this parabola. How do you know that this is the vertex?

      ($x = \frac{(-5)+(-1)}{2} = -3$*,* $y = (-3)^2 + 6(-3) + 5 = -4$*, so vertex is* $(-3, -4)$*.)*

      (Since the parabola is symmetric about the vertex, it must be equidistant from the two x-intercepts. So the x-value of the vertex will be halfway between the two x-intercepts, or the average of the them. Once we have the x-value, we solve for the y-value by plugging the x-value into the equation and solving for y.)

   e. Does this parabola open up or open down? (Is it "smiling" or "frowning"?) Explain how you know.

      (It opens up because the coefficient in front of the x^2 *term is positive, so it is "happy" and "smiling." Alternatively, both of the factors,* $x + 5$ *and* $x + 1$*, have positive slopes, and so the product is positive, or "happy" and so the parabola is "smiling.")*

   f. Now you are ready to graph the equation $y = x^2+6x+5$, or equivalently, $y = (x+5)(x+1)$. Go for it! Be sure to include the x-intercepts, y-intercept, and vertex in your graph.

DON'T ERASE YOUR WORK FROM PROBLEM 2 YET!!! For now, keep it while you do this next problem...

3. Now, suppose the equation is multiplied throughout by 2, so that it becomes $y = 2x^2+12x+10$.

   a. What will be the factored form for this equation?

      (It is the same, but you factor out the "2" first, called the "Greatest Common Factor" or "GCF", and then repeat the steps from earlier to obtain $y = 2(x+5)(x+1)$.)

   b. Will the x-intercepts change?

      (No. $x = -5$ and $x = -1$ will still make $y = 0$ so the x-intercepts are the same.)

   c. How will the y-intercept change?

      (It will be twice as much as before, since the equation is multiplied by 2. So $y = 10$ is the y-intercept now.)

   d. What will be the vertex now?

      The x-value of the vertex will still be midway between the same x-intercepts, so it doesn't change and is still $x = -3$. But the y-value will now by twice as great, so it will be $y = -8$. So the vertex is $(-3, -8)$. Also, we could use the same method as in problem 2 to find that x=-3 and then plug this x-value into the new equation to get $y = -8$.)

   e. Will the parabola open up or down now? Explain why.

      (Multiplying by 2 doesn't change the sign, it is still positive or "happy", so it still will open up.)

   f. Sketch the graph $y = 2x^2 + 12x + 10$.

STILL KEEP YOUR WORK FROM PROBLEM 2. DON'T ERASE IT! BUT YOU CAN ERASE YOUR LATEST WORK FROM PROBLEM 3...

4. Now take your earlier equation and multiply it through by a factor of "-1" so that it becomes $y = -x^2 - 6x - 5$.

   a. What will be the factored form for this equation?

      (Start by pulling out the GCF of "-1", and then factor the remaining "$x^2 + 6x + 5$", so that you get $y = -(x+5)(x+1)$.)

   b. What will be the x-intercepts? y-intercept? vertex? How do they compare to what you found for problem 2? Will this parabola open up or down now?

      (x-intercepts haven't changed. y-intercept will now be the negative of the earlier problem, so $y = -5$. Vertex will have same x-value but the y-value will be $-(-4) = +4$. Parabola will open down now because the coefficient in front of the x^2 term is negative, or alternatively, because (-)(+)(+) is negative, so the parabola will open down or "frown.")

   c. Graph the equation $y = -x^2 - 6x - 5$.

YOU CAN ERASE ALL OF YOUR EARLIER WORK NOW. WE'LL TRY SOME NEW PROBLEMS NOW!

5. Here are some more to practice. If short on time, skip some or all of these...

   a. Factor the equation $y = 3x^2 - 6x - 24$. Hint: If the coefficient in front of the x^2 term is not "1", then you'll need to pull out a GCF first.

      Find the x-intercepts, y-intercept, vertex, and determine whether the parabola opens up or down.

   b. Repeat with the equation $y = x^2 - 8x + 12$.

   c. This one is a little harder! Repeat with the equation $y = x^2 - 6x + 9$.

      (The factored form is $y = (x-3)(x-3)$, so there is only <u>one</u> x-intercept at x=3, and this is also the vertex, (3,0). y-intercept is $y = 9$, and the parabola opens up.)

6. Let's try working "backwards" now. That is, we'll sketch a parabola with certain features and determine the quadratic equation...

   a. On an xy-axis, sketch a parabola with (i) a vertex at $(3, -1)$, (ii) opening up, (iii) x-intercepts at $x = 2$ and $x = 4$, and (iv) y-intercept of $y = 8$.

      i. Using the x-intercepts and that the parabola opens up, write down the factored form form of the equation, $y = (x + \_\_)(x + \_\_)$. Now, expand this equation to give it in standard form.

      ii. To make sure your equation is correct, confirm that the y-intercept for your equation is $y = 8$ and that the vertex is $(3, -1)$.

   b. Now, sketch a parabola with the same x-intercepts as in part (a) of $x = 2$ and $x = 4$, but opening down instead so the vertex becomes $(3, 1)$ and the y-intercept becomes $y = -8$.

      How does this change the equation?

      (The factored form will now have a GCF of "-1" in front of it, so $y = -(x-2)(x-4)$ and so $y = -x^2 + 6x - 8$.)

   c. A little harder! Find the equation for a parabola with (i) *one* x-intercept of $x = -1$, (ii) a vertex at $(-1, 0)$, (iii) opening down, and (iv) a y-intercept of $y = -1$.

7. This next problem looks easy, but turns out to be MUCH harder...

   Write down the equation $y = 4x^2 - 8x + 3$.

   Can you tell what the y-intercept will be? (Hint: Set $x = 0$ and solve for y.)

   Can you tell whether the parabola will open up or down? (Hint: Is the coefficient in front of the x^2 term positive or negative?)

   Now, to determine the x-intercepts (and from those, the vertex), our trick (thus far) has been to rewrite the equation in factored form. Can you write this equation in factored form? (Hint: With the coefficient of "4" in front of the x^2 term, you would need o pull out a GCF of 4, but this can't be done because of the "+3" at the end of the equation.)

   Using our earlier trick of rewriting the equation in factored form *fails*! We are stuck, unless we find a new technique for finding the x-intercepts.

   Fortunately, we have one last trick up our sleeves for finding x-intercepts when the equation cannot be factored...

THE QUADRATIC FORMULA:
If the standard form of the equation is $y = ax^2 + bx + c$, then the x-intercepts will be

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

So, going back to our equation, $y = 4x^2 - 8x + 3$, what are the corresponding values for a, b, and c?

Now, use the quadratic formula to find the two x-intercepts.

$$x = \frac{-8 \pm \sqrt{(-8)^2 - 4(4)(3)}}{2(4)} = \frac{-8 \pm \sqrt{64 - 48}}{8} = \frac{-8 \pm \sqrt{16}}{8} = \frac{-8 \pm 4}{8}$$

(So $x = (-8 - 4)/8 = -12/8 = -3/2$ and $x = (-8 + 4)/8 = -4/8 = -1/2$).

What will the vertex be?

(x-value will be midpoint between the x-intercepts, so vertex has $x = -1$. Plugging this into the equation, $y = 4(-1)^2 - 8(-1) + 3 = 4 + 8 + 3 = 15$. The vertex is $(-1, 15)$.

8. More practice using the Quadratic Formula... For each equation, use the quadratic formula to solve for the x-intercepts.

   a. $y = 8x^2 + 6x - 2$

   b. $y = 2x^2 - 5x + 2$

   c. $y = x^2 + 6x + 5$.

      Note that this was the first equation we looked at together which we wrote in factored form as $y = (x + 5)(x + 1)$. And then we found the x-intercepts to be $x = -5$ and $x = -1$... the same x-intercepts that the quadratic formula gave us!

      So the quadratic formula will find the x-intercepts for *any* standard form equation, even one that can be written in factored form. But it is the *only* way to find the x-intercepts if the equation cannot be rewritten in factored form!